Unknotting Numbers of the First 12965 Knots

Alanna McNulty | Kindeep Singh Kargil | Liam Yethon | Martin Boote | Sheridan Houghten | Thomas Wolf

Brock University

Introduction

Work reported on this poster is part of larger project with the goal to create an interactive laboratory for knots and an online course for knot theory [1].
A knot diagram is the 2-dimensional projection of a non-intersecting closed curve in 3 dimensional space. If two diagrams can be deformed into each other by stretching, shortening or shifting the line without cutting it, then both diagrams represent the same mathematical knot.
One characteristic feature (invariant) of a knot is their unknotting number. This is the minimal number of times that the knot line needs to be cut and another part of the knot line be passed through the cut before the cut is sealed again in order to reach the so-called unknot (circle). The same unknotting number is obtaind when investigating all (infinitely many) diagrams of this knot by switching any crossing in this diagram (swapping the under- and overpassing strand in this crossing) and arbitrarily deforming the diagram and repeating this procedure to find the minimal number of such switches to reach the unknot.

Procedure

In our approach, for each one of the 59937 knots which admit a diagram with only up to 14 crossings we recursively applied deformations (so called flype and pass moves) that preserve the minimal number of crossings and in this way generated a multi million entry database of diagrams each simplified to the minimal number of crossings. Each diagram is represented through its Dowker-Thistlethwaite encoding (DT). The purpose is
(1) to get for each knot as many as possible minimal crossing diagrams, and
(2) to identify the mathematical knot from a given minimal crossing diagram.
For all 12965 knots with up to 13 crossings we use purpose (1) to investigate all their diagrams, each diagram by switching all crossings in it and after each switch maximally simplifying the diagram and use (2) to identify the knot and continue this search recursively. The leaves in this search tree are reached when unknots result. The unknotting number of a knot is taken to be the minimum total number of switches needed to reach the unknot from trying all diagrams and in each diagram trying all switches.
The whole procedure is complicated by the fact that after switching a crossing and simplifying the diagram a knot may result which is the sum of prime knots which are investigated individually. We apply the unproven but commonly accepted hypothesis that the unknotting number of a sum of knots is the sum of the unknotting numbers of its prime knots.

By starting the investigation with knots of lowest crossing numbers, the database is applied in the process of its own extension. As another speed up measure the depth-first tree search is also stopped at any level if there is no chance for the currently investigated branch to reach a lower unknotting number.

Results

Because we investigate knots only through their minimal crossing diagrams, and because of the mentioned handling of sums of prime knots, strictly speaking, our results are only upper bounds for the unknotting numbers. But our computations for all knots with up to 12 crossings give exactly the same upper bounds for unknotting numbers which were found by others (Mark Brittenham, Slavik Jablan, Radmila Sazdanovic, see [2] for references).
Most of these upper bound values are equal to lower bounds derived from theoretical considerations and are therefore exact.
Unknotting numbers for knots with > 12 crossings have not been published before and can now be downloaded from [3] for knots with ≤ 13 crossings.

Apart from these new numbers we have two more results.

A) On page 61 in [4] unsolved question 6 asks whether the unknotting number $u(K)$ of any knot can be realized by changing a single crossing in a minimal crossing diagram, re-arranging the resulting diagram to have minimal crossing number, changing another single crossing, rearranging to a minimal crossing diagram, etc. $u(K)$ times.
We found knots which have at least one minimal crossing diagram that has no simplifying (unknotting number reducing) crossing. Up to 13 crossings these are: $11n_{64}$, $11n_{138}$, $12n_{47}$, $12n_{258}$, $12n_{523}$, $12n_{723}$, $13n_{30}$, $13n_{45}$, $13n_{80}$, $13n_{221}$, $13n_{436}$, $13n_{447}$, $13n_{463}$, $13n_{636}$, $13n_{697}$, $13n_{700}$, $13n_{701}$, $13n_{779}$, $13n_{780}$, $13n_{1167}$, $13n_{1188}$, $13n_{2804}$, $13n_{2809}$, $13n_{2907}$, $13n_{2960}$, $13n_{3033}$, $13n_{3070}$, $13n_{3108}$, $13n_{3249}$, $13n_{3468}$, $13n_{3547}$, $13n_{3589}$, $13n_{4025}$, $13n_{4237}$, $13n_{4330}$, $13n_{4702}$, $13n_{4807}$, $13n_{4907}$.

Proof for $11n_{64}$:
Knot $11n_{64}$ has the unknotting number 2 (see [2]) and has a minimal diagram with the DT encoding: (6 -12 16 22 20 -4 18 10 2 14 8) (as can be verified with the KnotFinder in [2]). By changing the sign of the 13 numbers of this encoding individually one obtains the encodings of the 13 knots $9_{43}, 9_8, 3_1 + 6_1, 9_{43}, 9_{43}, 9_8, 8_4, 8_4, 9_8, 8_4, 9_{43}$ each of which has an unknotting number of 2 (see [2]). Therefore by switching any one crossing of this minimal diagram of $11n_{64}$ one will not lower the unknotting number. This answers the question by giving a counter example. On the other hand we do not know an example of a knot for which *none* of its minimal crossings diagrams has a simplifying move.

B) By performing a crossing switch in a minimal crossing diagram the crossing number can not increase after simplification. On the other hand, by performing unknotting number many switches the unknot is reached, so typically the crossing number reduces during the simplification that follows a switch. In our computations we noted very rare cases of unknotting switches not reducing the crossing number. As far as we know this has not been reported in the literature before.

Knot 10_{139} with unknotting number 4 has a diagram with DT code (6 10 14 -16 2 -18 4 -20 -8 -12). Switching a crossing (corresponding to a sign change in one of the numbers in the DE) gives a diagram with DT code (6 10 14 -16 2 -18 4 20 -8 -12) which is a diagram of knot 10_{161} which still has crossing number 10 but the unknotting number is now 3. Furthermore, this knot has a diagram with DT code (6 10 14 -16 4 -18 2 -20 -12 -8) which has a switch giving DT code (-6 10 14 -16 4 -18 2 -20 -12 -8) which encodes a diagram of knot 10_{145} which has unknotting number 2 but amazingly, still has crossing number 10.
A consequence of this finding is that one can not cut the tree search short just because a switch did not lower the crossing number of the knot.
The only other examples of knots with crossing number ≤ 11 with that property are knots $11n_{116}, 11n_{118}, 11n_{135}$ with unknotting number 2 which all have a diagram that has at least one crossing that gives after switching a new knot (knots $11n_{111}, 11n_{84}, 11n_{111}$) with unknotting number 1 but same crossing number 11. Among the 2176 knots with crossing number 12 there are 43 knots that have a diagram with a simplifying crossing that preserves the crossing number.
Currently (July 2019) calculations are underway to compute minimal crossing diagrams for all knots with up to 15 crossings and unknotting numbers for all knots with up to 14 crossings.

Display

Below, one diagram for each knot with up to 10 crossings is shown in a colour that corresponds to it's unknotting number. Colours: 1 2 3 4 . Simplifying switches for each knot are highlighted, their colour corresponds to how much the crossing number is reduced by that switch. Colours: 0 1 2 3 4 5 6 7 8 9 10 . The numbers in the caption above each knot have the following meaning. For example, 5_1 : 2 1 5 5 says that knot 5_1 has unknotting number 2, 1 minimal crossing diagram is known and has been investigated, from all the investigated diagrams the minimum number of simplifying switches in any one of them is 5 and the maximum number of simplifying switches is also 5.

References

[1] T. Wolf, "A Knot Workbench", https://cariboutests.com/games/knots/AsciiKnots.tar.gz
[2] J. C. Cha and C. Livingston, KnotInfo: Table of Knot Invariants, https://www.indiana.edu/~knotinfo especially https://www.indiana.edu/~knotinfo/descriptions/unknotting_number.html
[3] "Upper Bounds for Unknotting Numbers of Knots with Crossing Number up to 13", https://cariboutests.com/games/knots/unknotting3-13.txt
[4] Colin C. Adams, The knot book : an elementary introduction to the mathematical theory of knots, originally published by W.H. Freeman and Company, reprinted with corrections in 2004 by the American Mathematical Society, ISBN 0-8218-3678-1